CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2023, Exam 3

1. Define the following notions and provide a non-trivial example (2+2 points each).

   - L^p function for $p \in [1, \infty)$.
   - Essential supremum of a function.
   - Stronger and weaker topologies.
   - Topological subspace.
   - Compact set (in a general topological space).

   See the book.

2. State and prove *three* out of the following four theorems (10 points each).

   - Characterization of Lebesgue measurable sets (Prop. 3.2.3).
   - Lebesgue Dominated Convergence Theorem (for non-negative functions, Thm. 3.5.2).
   - Properties of compact sets.
   - The Weierstrass Theorem,

   See the book.

3. Let $G \subset \mathbb{R}^n$ be an open set. Prove that the family

$$\{F \subset \mathbb{R}^m \ : \ G \times F \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^m)\}$$

   is a σ-algebra in $\mathbb{R}^m$ (10 points).

   Solution: For any open set $F \subset \mathbb{R}^m$, Cartesian product $G \times F$ is open in $\mathbb{R}^n \times \mathbb{R}^m$. Thus, the family contains all open sets and, therefore, is not empty. The properties of σ-algebra follow then from simple identities,

$$G \times F' = G \times (\mathbb{R}^m - F) = G \times \mathbb{R}^m - G \times F$$

$$G \times \bigcup_n F_n = \bigcup_n (G \times F_n)$$

4. An integration exercise. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary *unbounded* open set, and let $f \in L^1(\Omega)$. Prove that

$$\int_{\Omega - B(0,n)} f(x)\,dx \to 0 \quad \text{as } n \to \infty\,.$$

where $B(0,n)$ denotes the ball centered at 0 with radius n. *Hint: Recall the Lebesgue Dominated Convergence Theorem* (10 points).

Solution:
Consider

$$f_n(x) := \begin{cases} 0 & x \in \Omega \cap \bar{B}(0,n) \\ f(x) & x \in \Omega - B(0,n) \end{cases}$$

Obviously, $f_n(x) \to 0$ as $n \to \infty$, and $|f_n(x)| \leq |f(x)|$, so $|f(x)|$ provides a dominating function. By the Lebesque Theorem,

$$\int_{\Omega - B(0,n)} f(x)\,dx = \int_{\Omega} f_n(x)\,dx \to 0\,.$$

5. Let X be an arbitrary topological space and $A \subset X$ an arbitrary set. Show that $\mathrm{int}A$ is the *largest open subset* of set A, and that closure $\overline{A}$ is the *smallest closed superset* of A (10 points).

Solution:
Let G be an open subset of A. Then $G = \mathrm{int}G \subset \mathrm{int}A$. Consequently,

$$\bigcup \{G \text{ open } \subset A\}$$

is the largest open subset of A and it is contained in $\mathrm{int}A$. As the $\mathrm{int}A$ is itself an open subset of A, the two sets must be equal.

By the same argument,

$$\overline{A} = \bigcap \{F \text{ closed } \supset A\}$$

is the smallest closed set containing set A.

6. We say that a topology has been introduced in a set X through the *operation of closure*, if we have introduced operation (of taking closure)

$$\mathcal{P}(X) \ni A \to \mathrm{cl}A \in \mathcal{P}(X) \quad \text{with} \quad A \subset \mathrm{cl}A$$

that satisfies the following four properties:

(i) $\mathrm{cl}\emptyset = \emptyset$

(ii) $A \subset B$ implies $\mathrm{cl}A \subset \mathrm{cl}B$

(iii) $\mathrm{cl}(\mathrm{cl}A) = \mathrm{cl}A$

(iv) $\mathrm{cl}(A \cup B) = \mathrm{cl}A \cup \mathrm{cl}B$

Sets F such that $\mathrm{cl}F = F$ are identified then as *closed sets*.

(a) Prove that the closed sets defined in this way, satisfy the usual properties of closed sets (empty set and the whole space are closed, intersections of arbitrary families, and unions of finite families of closed sets are closed) (7 points).

(b) Define *open sets* $\mathcal{X}$ by taking complements of *closed sets*. Notice that the duality argument implies that family $\mathcal{X}$ satisfies the axioms for the open sets. Use then family $\mathcal{X}$ to introduce a topology (through open sets) in X. Consider next the corresponding closure operation $A \to \overline{A}$ with respect to the new topology. Prove then that the original and the new operations of taking the closure coincide with each other, i.e.,

$$\mathrm{cl}A = \overline{A}$$

for every set A (6 points).

(c) Conversely, assume that a topology was introduced by open sets $\mathcal{X}$. The corresponding operation of closure satisfies then properties listed above and can be used to introduce a (potentially different) topology and corresponding (potentially different) open sets $\mathcal{X}'$. Prove that families $\mathcal{X}$ and $\mathcal{X}'$ must be identical (7 points).

(total 20 points).

Solution:

(a) By assumption, $X \subset \mathrm{cl}X \subset X$ so $\mathrm{cl}X = X$. The empty set is closed by axiom (i). Let $F_\iota, \iota \in I$, be now an arbitrary family of sets such that $\mathrm{cl}F_\iota = F_\iota$. By assumption,

$$\mathrm{cl}\bigcap_{\iota \in I} F_\iota \supset \bigcap_{\iota \in I} F_\iota = \bigcap_{\iota \in I} \mathrm{cl}F_\iota$$

Conversely,

$$F_\iota \supset \bigcap_{\kappa \in I} F_\kappa, \quad \forall \iota \in I$$

Axiom (ii) implies that,

$$F_\iota \supset \mathrm{cl}\bigcap_{\kappa \in I} F_\kappa, \quad \forall \iota \in I$$

and, consequently,

$$\bigcap_{\iota \in I} F_\iota \supset \mathrm{cl}\bigcap_{\kappa \in I} F_\kappa$$

Finally, by induction, axiom (iv) implies that closure of a union of a finite family of sets is equal to the union of their closures.

(b) Let A be an arbitrary set. Recall that $\overline{A}$ is the smallest closed set that contains A. By axiom (iii), $\mathrm{cl}A$ is closed and it contains A, so $\overline{A} \supset \mathrm{cl}A$. On the other side, $A \subset \mathrm{cl}A$ and the fact that $\mathrm{cl}A$ is closed imply $\overline{A} \subset \overline{\mathrm{cl}A} = \mathrm{cl}A$.

(c) This is trivial. Closed sets F in both families are identified through the same property: $\overline{F} = \mathrm{cl}F = F$.