ASE 380P 2-ANALYTICAL METHODS II
EM386L MATHEMATICAL METHODS IN APPLIED MECHANICS II
CSE 386L MATHEMATICAL METHODS IN APPLIED ENGINEERING AND SCIENCES

Exam 1. Monday, Feb 27, 2012

1. (a) State the Implicit Function Theorem (5 points).

   (b) Expand $z(x, y)$ about $(0, 0)$, through terms of second order for the implicitly defined function,
   $$x^2 + y^2 + 2z^2 = 4, \quad z \geq 0$$
   (15 points).

   (c) Denoting the second-order approximation by $z_{approx}(x, y)$, sketch the original and approximating surfaces $z(x, y)$ and $z_{approx}(x, y)$, respectively (5 points).

2. (a) Consider standard spherical coordinates,
   $$\begin{cases} x = r \sin\psi \cos\theta \\ y = r \sin\psi \sin\theta \\ z = r \cos\psi \end{cases}$$
   Draw a picture representing the coordinates and the corresponding unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\psi, \boldsymbol{e}_\theta$ (5 points).

   (b) Assume that (r, ψ, θ) are functions of time t. Derive the formula for the velocity and acceleration vector in the curvilinear system of coordinates (10 points).

   (c) Use the formulas to compute the acceleration vector for a point moving on a sphere of radius R,
   $$r = R, \quad \psi = \frac{\pi}{2}t, \quad \theta = \pi t$$
   at time $t = 1$. Compute the tangential and normal acceleration at that moment (10 points).

3. (a) Formulate the Stokes' Theorem (5 points).

   (b) Verify the Stokes' Theorem by computing the necessary surface and line integrals for the following vector field:
   $$\boldsymbol{v} = (x^3y, y^2 + 8, 0)$$

and the hemispherical cap:

$$z = \sqrt{4 - x^2 - y^2}$$

(20 points).

4. (a) Define the *geodesics* (5 points).

   (b) Find the geodesics for the (lateral surface of the) cylinder $x^2 + y^2 = R^2$. *Hint:* Use cylindrical coordinates to formulate the variational problem and solve it (20 points).

1a)

Given:

$$F: \mathbb{R}^n \times \mathbb{R}^m \ni (x, y) \longrightarrow F(x,y) \in \mathbb{R}^m$$

$$x_0 \in \mathbb{R}^n, \; y_0 \in \mathbb{R}^m, \quad F(x_0, y_0) = 0$$

$$\det\left(\frac{\partial F}{\partial y}\right) \neq 0$$

There exist then:

a neighborhood of x_0, e.g. ball $B(x_0, \varepsilon_1)$

a neighborhood of y_0, e.g. ball $B(y_0, \varepsilon_2)$

function $B(x_0, \varepsilon_1) \ni x \longrightarrow y = f(x) \in B(y_0, \varepsilon_2)$

such that

$$F(x, f(x)) = 0 \quad \forall x \in B(x_0, \varepsilon) \qquad (*)$$

By differentiating $(*)$ in x, we can compute the derivatives of the implicitly defined function

$$\frac{\partial F_k}{\partial x_j} + \frac{\partial F_k}{\partial y_i}\frac{\partial y_i}{\partial x_j} = 0 \quad \Rightarrow \quad \frac{\partial y_i}{\partial x_j} = -\left(\frac{\partial F_k}{\partial y_i}\right)^{-1}\frac{\partial F_k}{\partial x_j}$$

1b)

$$x^2 + y^2 + 2z^2 = 4 \qquad \Rightarrow \quad z = \sqrt{2}$$

$$2x + 4z\frac{\partial z}{\partial x} = 0 \qquad \Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{2x}{4z} = 0$$

Analogously $\dfrac{\partial z}{\partial y} = 0$

$$2 + 4\left(\frac{\partial z}{\partial x}\right)^2 + 4z\frac{\partial^2 z}{\partial x^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 z}{\partial x^2} = -\frac{1}{2z} = -\frac{\sqrt{2}}{4}$$

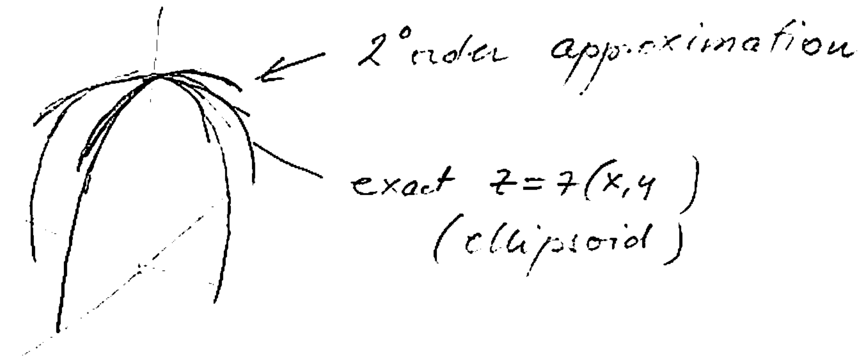
$$4\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + 4z\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \Rightarrow \quad \frac{\partial^2 z}{\partial x \partial y} = 0$$

Analogously $\dfrac{\partial^2 z}{\partial y^2} = 0$

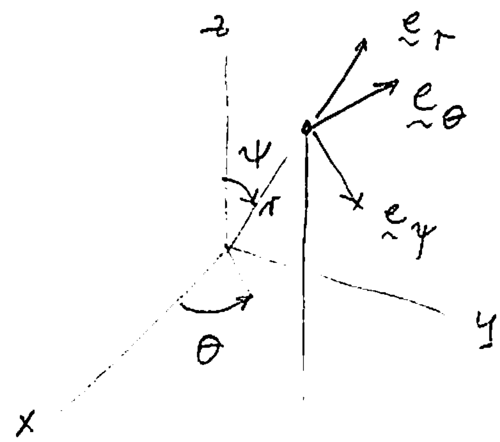
Second order approximation

$$z(x,y) = z(0,0) + \frac{\partial z}{\partial x}(0,0)\,x + \ldots + \frac{1}{2}\frac{\partial^2 z}{\partial y^2}(0,0)\,y^2$$

$$= \sqrt{2} - \frac{\sqrt{2}}{8}x^2 - \frac{\sqrt{2}}{8}y^2$$



2a)



$$\underset{\sim}{a}_r = \frac{\partial \underset{\sim}{r}}{\partial r} = (\sin\psi\cos\theta,\ \sin\psi\sin\theta,\ \cos\psi) \qquad |\underset{\sim}{a}_r| = 1$$

$$\underset{\sim}{a}_\psi = \frac{\partial \underset{\sim}{r}}{\partial \psi} = (r\cos\psi\cos\theta,\ r\cos\psi\sin\theta,\ -r\sin\psi) \qquad |\underset{\sim}{a}_\psi| = r$$

$$\underset{\sim}{a}_\theta = \frac{\partial \underset{\sim}{r}}{\partial \theta} = (-r\sin\psi\sin\theta,\ r\sin\psi\cos\theta,\ 0) \qquad |\underset{\sim}{a}_\theta| = r\sin\psi$$

$$\underset{\sim}{e}_r = \frac{\underset{\sim}{a}_r}{|\underset{\sim}{a}_r|} = \underset{\sim}{a}_r$$

$$\underset{\sim}{e}_\psi = \frac{\underset{\sim}{a}_\psi}{|\underset{\sim}{a}_\psi|} = (\cos\psi\cos\theta,\ \cos\psi\sin\theta,\ -\sin\psi)$$

$$\underset{\sim}{e}_\theta = \frac{\hat{\underset{\sim}{a}}_\theta}{|\underset{\sim}{a}_\theta|} = (-\sin\theta,\ \cos\theta,\ 0)$$

2b)
$$\frac{\partial \mathbf{e}_r}{\partial \psi} = (\cos\psi\cos\theta,\ \cos\psi\sin\theta,\ -\sin\psi) = \mathbf{e}_\psi$$

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = (-\sin\psi\sin\theta,\ \sin\psi\cos\theta,\ 0) = \sin\psi\ \mathbf{e}_\theta$$

$$\frac{\partial \mathbf{e}_\psi}{\partial \psi} = (-\sin\psi\cos\theta,\ -\sin\psi\sin\theta,\ -\cos\psi) = -\mathbf{e}_r$$

$$\frac{\partial \mathbf{e}_\psi}{\partial \theta} = (-\cos\psi\sin\theta,\ \cos\psi\cos\theta,\ 0) = \cos\psi\ \mathbf{e}_\theta$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \psi} = 0$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = (-\cos\theta,\ -\sin\theta,\ 0) = -[\sin\psi\ \mathbf{e}_r + \cos\psi\ \mathbf{e}_\psi]$$

Position: $\mathbf{r} = r\,\mathbf{e}_r$

Velocity: $\mathbf{v} = \dot{\mathbf{r}} = \dot r\,\mathbf{e}_r + r\left(\frac{\partial \mathbf{e}_r}{\partial\psi}\dot\psi + \frac{\partial \mathbf{e}_r}{\partial\theta}\dot\theta\right)$

$$= \dot r\,\mathbf{e}_r + r\dot\psi\,\mathbf{e}_\psi + r\sin\psi\,\dot\theta\,\mathbf{e}_\theta$$

Acceleration:

$$\mathbf{a} = \ddot r\,\mathbf{e}_r + \dot r\Big(\underbrace{\frac{\partial \mathbf{e}_r}{\partial\psi}}_{=\mathbf{e}_\psi}\dot\psi + \underbrace{\frac{\partial \mathbf{e}_r}{\partial\theta}}_{=\sin\psi\,\mathbf{e}_\theta}\dot\theta\Big)$$

$$+ \dot r\dot\psi\,\mathbf{e}_\psi + r\ddot\psi\,\mathbf{e}_\psi + r\dot\psi\Big(\underbrace{\frac{\partial \mathbf{e}_\psi}{\partial\psi}}_{=-\mathbf{e}_r}\dot\psi + \underbrace{\frac{\partial \mathbf{e}_\psi}{\partial\theta}}_{=\cos\psi\,\mathbf{e}_\theta}\dot\theta\Big)$$

$$+ \dot r\sin\psi\,\dot\theta\,\mathbf{e}_\theta + r\cos\psi\,\dot\psi\,\dot\theta\,\mathbf{e}_\theta + r\sin\psi\,\ddot\theta\,\mathbf{e}_\theta$$

$$+ r\sin\psi\,\dot\theta\Big(\underbrace{\frac{\partial \mathbf{e}_\theta}{\partial\theta}}_{=-(\sin\psi\,\mathbf{e}_r + \cos\psi\,\mathbf{e}_\psi)}\dot\theta\Big)$$

so :

$$\begin{cases} v_r = \dot{r} \\ v_\psi = r\dot{\psi} \\ v_\theta = r\sin\psi\,\dot{\theta} \end{cases}$$

$$a_r = \ddot{r} - r\dot{\psi}^2 - r\sin^2\psi\,\dot{\theta}^2$$

$$a_\psi = 2\dot{r}\dot{\psi} + r\ddot{\psi} - r\sin\psi\cos\psi\,\dot{\theta}^2$$

$$a_\theta = 2\dot{r}\dot{\theta}\sin\psi + 2r\dot{\psi}\cos\psi\,\dot{\theta} + r\sin\psi\,\ddot{\theta}$$

2c) $r = R \Rightarrow \dot{r} = \ddot{r} = 0$

$\psi = \frac{\pi}{2}t \Rightarrow \dot{\psi} = \frac{\pi}{2},\ \ddot{\psi} = 0$

$\theta = \pi t \Rightarrow \dot{\theta} = \pi,\ \ddot{\theta} = 0$

so,

$$a_r = -R\left(\frac{\pi}{2}\right)^2 - R\pi^2\sin^2\psi$$

$$a_\psi = -R\sin\psi\cos\psi\,\pi^2$$

$$a_\theta = 2R\,\frac{\pi}{2}\cos\psi\,\pi$$

$$v_r = 0$$
$$v_\psi = R\frac{\pi}{2}$$
$$v_\theta = R\pi\sin\psi$$

$t=1 \Rightarrow \psi = \frac{\pi}{2} \Rightarrow \cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1$

$a_r = -\frac{5}{4}R\pi^2,\quad a_\psi = 0,\quad a_\theta = 0$

$v_r = 0,\quad v_\psi = \frac{1}{2}R\pi,\quad v_\theta = R\pi$

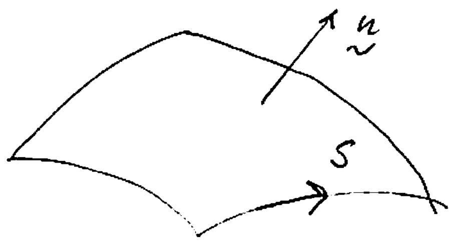
tangential acceleration :

$$\underset{\sim}{a}_t = \left(\underset{\sim}{a} \cdot \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}\right) \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \underset{\sim}{0}$$

normal acceleration :

$$\underset{\sim}{a}_n = \underset{\sim}{a} - \underset{\sim}{a}_t = \underset{\sim}{a} .$$

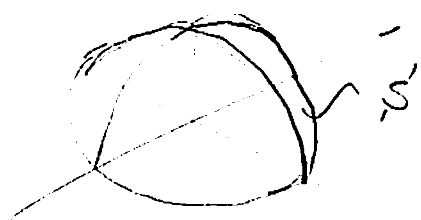
3a)



$$\underset{\sim}{E} : S \longrightarrow \mathbb{R}^3$$

$$\int_S \operatorname{curl} \underset{\sim}{E} \cdot \underset{\sim}{n} = \int_{\partial S} \underset{\sim}{E} \cdot d\underset{\sim}{r}$$

3b)



Line integral :

Parametrization :

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \\ z = 0 \end{cases} \qquad \theta \in (-\pi, \pi)$$

$$\begin{cases} \dfrac{dx}{d\theta} = -2\sin\theta \\ \dfrac{dy}{d\theta} = 2\cos\theta \\ \dfrac{dz}{d\theta} = 0 \end{cases}$$

$$\int_{\partial S} \underset{\sim}{E} \cdot d\underset{\sim}{r} = \int_0^{2\pi} \left(\underset{\sim}{E} \cdot \frac{dr}{d\theta} \right) d\theta$$

$$= \int_0^{2\pi} \left\{ 8\cos^3\theta \sin\theta \, (-2\sin\theta) + (4\cos^2\theta + 8)\, 2\cos\theta \right\} d\theta$$

$$= -16 \int_0^{2\pi} \cos^3\theta \sin^2\theta \, d\theta + 8 \int_0^{2\pi} \cos^3\theta \, d\theta + 16 \int_0^{2\pi} \cos\theta \, d\theta$$

All integrals are zero!

$$\int_0^{2\pi} \cos\theta \, d\theta = \sin\theta \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \cos^3\theta \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} \cos^3\theta \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \, d\theta + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \cos^3\theta \, d\theta = 0$$

cancel out!

$$\int_0^{2\pi} \cos^3\theta \sin^2\theta \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} = 0$$

cancel out

Surface integral

$$\begin{array}{rcccl} & \underset{\sim}{\nabla} & (\partial_x, & \partial_y, & \partial_z) \\ \times & \underset{\sim}{v} & (x^3 y, & y^2 + 8, & 0) \\ \hline \text{curl } \underset{\sim}{v} & & (0, & 0, & -x^3) \end{array}$$

$$\int_S \text{curl } \underset{\sim}{v} \cdot \underset{\sim}{n} \, dS = -\int_S x^3 n_z \, dS = 0$$

(by symmetry argument again!)

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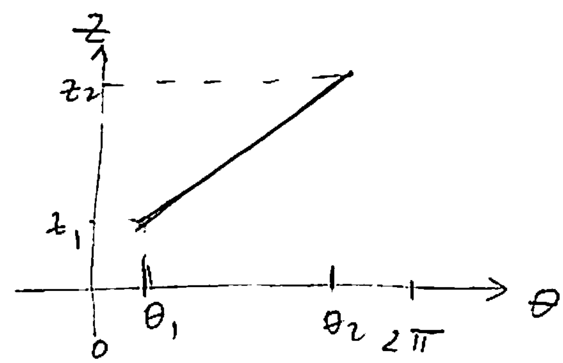
4a) Geodesics of a surface : curves on the surface connecting two points with a minimum length.

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases}$$



Look for the geodesics in the form $z = z(\theta)$

Then

$$\underset{\sim}{r}(\theta) = (R\cos\theta,\ R\sin\theta,\ z(\theta))$$

$$\frac{d\underset{\sim}{r}}{d\theta} = (-R\sin\theta,\ R\cos\theta,\ z')$$

$$\left|\frac{d\underset{\sim}{r}}{d\theta}\right| = \sqrt{R^2+(z')^2}$$

Length of a curve connecting points (θ_1, z_1), (θ_2, z_2) $(\theta_1 < \theta_2)$:

$$\int_{\theta_1}^{\theta_2} \left|\frac{d\underset{\sim}{r}}{d\theta}\right| d\theta = \int_{\theta_1}^{\theta_2} \underbrace{\sqrt{R^2+(z')^2}}_{L(z',\cancel{z},\cancel{\theta})}\, d\theta$$

Euler-Lagrange equ:

$$-\left[\frac{\cancel{2}z'}{\cancel{2}\sqrt{R^2+z'^2}}\right]' = 0$$

so $$\frac{z'}{\sqrt{R^2+z'^2}} = c = \text{const}$$

$$z'^2 = c^2(R^2 + z'^2)$$

$$z'^2(1-c^2) = c^2R^2$$

$$z'^2 = \frac{c^2R^2}{1-c^2} \Rightarrow \underline{z' = \text{const}}$$

Geodesics, must be a straight line in the θ-z plane.

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