CSE386L MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
Spring 22, Exam 1

1. 3D calculus. Consider cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

and a cylinder: $D : r < 1, |z| < 1$.

- Derive formulas for the gradient and divergence in the cylindrical system of coordinates.
- State Gauss Divergence Theorem.
- Verify the Gauss' Theorem for the field:

$$\boldsymbol{v} = r\boldsymbol{e}_r + r\theta\boldsymbol{e}_\theta + z\boldsymbol{e}_z$$

by computing the necessary volume and surface integrals.

(20 points)

Solution:

- The general formula for the gradient in any curvilinear system of coordinates $\boldsymbol{x} = \boldsymbol{x}(\xi_j)$ is:

$$\boldsymbol{\nabla} u = \frac{\partial u}{\partial \xi_j}\boldsymbol{a}^j$$

where $\boldsymbol{a}^j$ are the co-basis vectors. The basis vectors for cylindrical coordinates are:

$$\boldsymbol{a}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\cos\theta, \sin\theta, 0), \quad \boldsymbol{a}_\theta = \frac{\partial \boldsymbol{x}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0), \quad \boldsymbol{a}_z = \frac{\partial \boldsymbol{x}}{\partial z} = (0, 0, 1)\,.$$

The system is orthogonal, so determining the co-basis vectors reduces to scaling,

$$\boldsymbol{a}_r = \boldsymbol{e}_r \;\Rightarrow\; \boldsymbol{a}^r = \boldsymbol{e}_r \quad \boldsymbol{a}_\theta = r\boldsymbol{e}_\theta \;\Rightarrow\; \boldsymbol{a}^\theta = \frac{1}{r}\boldsymbol{e}_\theta \quad \boldsymbol{a}_z = \boldsymbol{e}_z \;\Rightarrow\; \boldsymbol{a}^z = \boldsymbol{e}_z\,.$$

The gradient thus is:

$$\boldsymbol{\nabla} u = \frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial u}{\partial z}\boldsymbol{e}_z\,,$$

Check quickly consistency of units for each term.

The fastest way to derive the formula for the divergence is by utilizing the integration by parts:

$$\int \boldsymbol{v}\boldsymbol{\nabla} u = -\int \operatorname{div} \boldsymbol{v} u + B.T.$$

Let $\boldsymbol{v} = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta + v_z \boldsymbol{e}_z$. We have,

$$\int\int\int v_r \frac{\partial u}{\partial r} + \frac{1}{r} v_\theta \frac{\partial u}{\partial \theta} + v_z \frac{\partial u}{\partial z}\, dr d\theta dz = -\int\int\int \underbrace{\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right)}_{=\operatorname{div} \boldsymbol{v}} r dr d\theta dz + B.T.$$

- **Gauss Divergence Theorem:**
  Let $\Omega \subset \mathbb{R}^3$ be a domain (= open and connected set), and let $\boldsymbol{v} \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \operatorname{div} \boldsymbol{v}\, dx = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n}\, dS$$

  where $\boldsymbol{n}$ is the outward normal unit vector to $\partial\Omega$.

- Verify the Gauss' Theorem for the field:

$$\boldsymbol{v} = r\boldsymbol{e}_r + r\theta \boldsymbol{e}_\theta + z\boldsymbol{e}_z$$

  by computing the necessary volume and surface integrals. The volume integral is:

$$\int \operatorname{div} \boldsymbol{v} = \int_0^1 \int_0^{2\pi} \int_{-1}^1 (2+1+1)\, r dr d\theta dz = 4\pi 1^2 2 = 8\pi\,.$$

  The integral over the lateral surface:

$$v_n = \boldsymbol{v} \cdot \boldsymbol{e}_r = r = 1 \quad \Rightarrow \quad \int_S v_n\, dS = \int_S dS = 4\pi\,.$$

  The integrals over the bottom and top faces are equal since $v_n = -z = 1$ on the bottom face, and $v_n = z = 1$ on the top face as well. The sum of the two integrals is thus $2\pi 1^2 = 2\pi$. The theorem does not verify: $8\pi \neq 6\pi$. Reason: field $\boldsymbol{v}$ is not even continuous over D. Drop the (discontinuous) θ component and everything checks out.

2. Jordan decomposition and systems of ODEs. Consider the matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

- Determine generalized eigenvectors of matrix A and the corresponding Jordan form.
- Use the Jordan form to determine general solution for the system of ODEs:

$$\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u}\,.$$

(20 points)

Solution: The matrix is upper triangular, so the terms on the diagonal are the eigenvalues, we have a single eigenvalue $\lambda = 1$, and a double eigenvalue $\lambda = 2$. Solving for the eigenvector corresponding to $\lambda = 1$,

$$(\boldsymbol{A} - 1\boldsymbol{I})\boldsymbol{x} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = (t, 0, 0)^T\,.$$

We can choose $t = 1$, getting $\boldsymbol{e}_1 = (1, 0, 0)^T$. Solving for eigenvectors corresponding to $\lambda = 2$,

$$(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = (t, t, 0)^T\,.$$

We have only one eigenvector. Solving for the corresponding generalized eigenvector in the chain:

$$(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = (u, u + \frac{1}{3}t, \frac{1}{3}t)^T\,.$$

We can choose $t = 1$, $u = 1$, getting $\boldsymbol{e}_2 = (1, 1, 0)^T$, $\boldsymbol{e}_3 = (1, \frac{4}{3}, \frac{1}{3})$.

By the Jordan Theorem, matrix $\boldsymbol{A}$ takes the following form in the eigenbasis.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus, seeking the solution to the system of ODEs in the eigenbasis,

$$\boldsymbol{x} = c_1(t)\boldsymbol{e}_1 + c_2(t)\boldsymbol{e}_2 + c_3(t)\boldsymbol{e}_3\,,$$

we obtain the following system of equations:

$$\begin{cases} \dot{c_1} &= c_1 \\ \dot{c_2} &= 2c_2 \\ \dot{c_3} &= c_2 + 2c_3 \end{cases}$$

This leads to: $c_1 = C_1 e^t$, $c_2 = C_2 e^{2t}$, $c_3 = (C_2 t + C_3)e^{2t}$ and the final formula for the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (C_2 t + C_3)e^{2t} \begin{pmatrix} 1 \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}.$$

3. A Sturm-Liouville problem. Consider the operator:

$$Au = -u'', \quad D(A) = \{u \in L^2(0,1) \, : u'' \in L^2(0,1), \quad u(0) = 0, \quad u(1) - 2u'(1) = 0\}\,.$$

- Use integration by parts to show (formally) that the operator is self-adjoint.
- Show that the eigenfunctions are of the form $\sin\sqrt{\lambda_n}x$, where the eigenvalues λ_n are solutions to the transcendental equation:

$$\tan\sqrt{\lambda} = 2\sqrt{\lambda}\,.$$

- Argue that

$$\lambda_n = (2n-1)^2\frac{\pi^2}{4} \quad \text{as } n \to \infty\,.$$

(20 points)

Solution:

- We have,

$$\begin{aligned}
\textstyle\int_0^1 (-u'')v\,dx &= \textstyle\int_0^1 u'v'\,dx - (u'v)|_0^1 \\
&= \textstyle\int_0^1 u(-v'')\,dx - (u'v)|_0^1 + uv'|_0^1 \\
&= \textstyle\int_0^1 u(-v'')\,dx - \frac{1}{2}u(1)v(1) + u'(0)v(0) + u(1)v'(1) \\
&= \textstyle\int_0^1 u(-v'')\,dx - \frac{1}{2}u(1)[v(1) - 2v'(1)] + u'(0)v(0) \\
&= \textstyle\int_0^1 u(-v'')\,dx\,,
\end{aligned}$$

  provided $v(0) = 0$ and $v(1) - 2v'(1) = 0$.

- The operator is self-adjoint and, actually, it is also positive definite. We have,

$$\int_0^1 (-u'')u\,dx = \int_0^1 (u'(x))^2\,dx - \underbrace{u'(1)}_{=\frac{1}{2}u(1)}\,u(1) + u'(0)\underbrace{u(0)}_{=0} = \int_0^1 (u'(x))^2\,dx - \frac{1}{2}(u(1))^2\,.$$

  It is not obvious that the sum above must be positive. We need to utilize the BC at 0,

$$(u(1))^2 = (\int_0^1 u'(s)\,ds)^2 \le \int_0^1 (u'(s))^2\,ds \qquad \text{(Cauchy-Schwarz inequality at work)},$$

  so

$$(-u'', u) = \|u'\|^2 - \frac{1}{2}|u(1)|^2 \ge \frac{1}{2}\|u'\|^2\,.$$

  Thus, if the left-hand side is zero, $u' = 0$, i.e., u is a constant and, by the BC $u(0) = 0$, it must be zero. Consequently, we know ahead of time that the eigenvalues are real and positive. This leads to:

$$u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\,.$$

BC at $x = 0$ implies that $A = 0$. BC at $x = 1$ leads to the equation:

$$0 = u(1) - 2u'(1) = B(\sin\sqrt{\lambda} - 2\sqrt{\lambda}\cos\sqrt{\lambda})$$

and, in turn, to:

$$\tan\sqrt{\lambda} = 2\sqrt{\lambda}\,.$$

- A picture representing both sides of the equation shows that we have an infinite sequence of eigenvalues. For large values of n, the intersection point of $2x$ with $\tan x$ gets close to the intersection between $2x$ and the asymptote for the n-th branch of $\tan x$, i.e., the line $x = (n - \frac{1}{2})\pi$. This gives,

$$\lambda_n \approx (n - \frac{1}{2})^2\pi^2 = (2n-1)^2\frac{\pi^2}{4}\,.$$

4. This is the correct rewrite of the original problem.
Another Sturm-Liouville problem. Consider the Sturm-Liouville operator,

$$Au = -(au')' + cu, \quad D(A) = \{u \in L^2(-l,l) \,:\, Au \in L^2(-l,l),\, u(-l) = u(l) = 0\}$$

where the diffusion and reaction coefficients are even functions, i.e.,

$$a(-x) = a(x), \quad c(-x) = c(x)\,.$$

Prove that if $(\lambda, u(x))$ is an eigenpair for operator A then so is $(\lambda, u(-x))$. Conclude that, If the eigenvector is neither even nor odd, the even and odd parts of function u ($\frac{1}{2}(u(x) + u(-x)), \frac{1}{2}(u(x) - u(-x))$) must be eigenvectors corresponding to λ as well. One can search then from the very beginning separately for even and odd eigenvectors which simplifies greatly the algebra. The eigenspace is then at least two-dimensional. Note that, if the original eigenvector is even (odd) to begin with, then the search for the odd (even) eigenvector will simply fail.

(20 points)

Solution: Define: $v(x) := u(-x)$. The chain formula implies that

$$\frac{dv}{dx}(x) = -\frac{du}{dx}(-x) \quad \text{and} \quad \frac{d^2v}{dx^2}(x) = -\frac{d^2u}{dx^2}(-x)\,,$$

Consequently,

$$\begin{aligned} [-(av')' + cv](x) &= [-av'' - a'v + cv](x) \\ &= [-au'' - a'u + cu](-x) = [-(au')' + cu](-x) \\ &= \lambda u(-x) = \lambda v(x)\,. \end{aligned}$$

The rest of conclusions follows.

5. Legendre polynomials. Consider the Legendre operator:

$$Au = -((1-x^2)u')', \quad D(A) := \{u \in L^2(-1,1) \, : \, Au \in L^2(-1,1)\} .$$

- Demonstrate (formally) that operator A is self-adjoint.
- Determine eigenpairs for the operator by seeking the eigenvectors in the form of their Taylor expansions at $x = 0$. ***Hint:*** $\lambda_n = n(n+1)$, $n = 0, 1, 2, \ldots$.

(20 points)

- The boundary terms are zero since $a(x) = (1-x^2)$ vanishes at the end points,

$$\int_{-1}^{1} (-((1-x^2)u')')v \, dx = \int_{-1}^{1} ((1-x^2)u'v' \, dx = \int_{-1}^{1} u(-((1-x^2)v')') \, dx .$$

- Without loosing any generality, we can assume $\lambda = \nu(\nu+1)$, $\nu \geq 0$. We begin by rewriting the Legendre equation in the form more suitable for the Frobenius method,

$$(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0 .$$

Note that any $x = \pm 1$ are regular singular points, and any other point $x_0 \in I$ is a regular point. We will expand around zero and seek solutions for integer values of $\nu =: n$.

$$\begin{aligned} y &= \sum_{k=0}^{\infty} c_k x^k & n(n+1)y &= n(n+1)\sum_{k=0}^{\infty} c_k x^k \\ y' &= \sum_{k=1}^{\infty} k c_k x^{k-1} & -2xy' &= -2\sum_{k=1}^{\infty} k c_k x^k \\ y'' &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k & -x^2 y'' &= -\sum_{k=2}^{\infty} k(k-1) c_k x^k \end{aligned}$$

Substituting into the equation, we get the relations:

$$\begin{aligned} &k = 0 \qquad 2c_2 + n(n+1)c_0 = 0 && \Rightarrow \quad c_2 = -\tfrac{n(n+1)}{2} c_0 \\ &k = 1 \qquad 6c_3 + (n-1)(n+2)c_1 = 0 && \Rightarrow \quad c_3 = -\tfrac{(n-1)(n+2)}{6} c_1 \\ &k > 1 \qquad (k+2)(k+1)c_{k+2} + [n(n+1) - k(k+1)]c_k = 0 && \Rightarrow \quad c_{k+2} = -\tfrac{n(n+1)-k(k+1)}{(k+2)(k+1)} c_k . \end{aligned}$$

We obtain thus two solutions corresponding to pairs $c_0 = 1, c_1 = 0$ and $c_0 = 0, c_1 = 1$,

$$\begin{aligned} y_1 &= 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots \\ y_2 &= x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots . \end{aligned}$$

Note that if n is even, the y_1 series at some point terminates, and y_1 is simply a polynomial. Similarly, if n is odd, the y_2 series terminates. Thus, for any natural number n, we have two solutions: a polynomial solution $P_n(x)$ and a second solution $Q_n(x)$ represented with an infinite series. Functions $P_n(x)$ are the *Legendre polynomials* or *Legendre functions of the first kind and degree* n, functions $Q_n(x)$ are *Legendre functions of the second kind and degree* n.