EM311M Notes

1 Kinematics of a particle

Definitions

position vector

$$\boldsymbol{r} = \boldsymbol{r}(t)$$

velocity vector

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt}$$

speed

$$|\boldsymbol{v}|$$

acceleration vector

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$$

tangential and normal acceleration vectors

$$\begin{aligned}\boldsymbol{a}_t &= \left(\boldsymbol{a} \cdot \frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right) \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \\ \boldsymbol{a}_n &= \boldsymbol{a} - \boldsymbol{a}_t\end{aligned}$$

Cartesian coordinates

$$\begin{aligned}\boldsymbol{r} &= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \\ \boldsymbol{v} &= \dot{\boldsymbol{r}} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})^{\cdot} = \underbrace{\dot{x}}_{v_x}\boldsymbol{i} + \underbrace{\dot{y}}_{v_y}\boldsymbol{j} + \underbrace{\dot{z}}_{v_z}\boldsymbol{k}\end{aligned}$$

So $a_x = \dot{v}_x = \ddot{x}, \ldots$.

Rectilinear motion

$$
\begin{aligned}
&\boldsymbol{r} = x\boldsymbol{i} \\
&s := x \\
&v := v_x = \dot{x} = \dot{s} \\
&a := a_x = \dot{v} = \ddot{s} \\
&a = \frac{dv}{dt},\ v = \frac{ds}{dt} \Rightarrow a\,ds = v\,dv
\end{aligned}
$$

Cylindrical coordinates

$$
\begin{cases}
x = r\cos\theta \\
y = r\sin\theta \\
z = z
\end{cases}
$$

cylindrical unit vectors

$$
\begin{aligned}
\boldsymbol{a}_r &= \frac{\partial \boldsymbol{r}}{\partial r} = (\cos\theta, \sin\theta, 0) & \boldsymbol{e}_r &= \frac{\boldsymbol{a}_r}{|\boldsymbol{a}_r|} = (\cos\theta, \sin\theta, 0) \\
\boldsymbol{a}_\theta &= \frac{\partial \boldsymbol{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0) & \boldsymbol{e}_\theta &= \frac{\boldsymbol{a}_\theta}{|\boldsymbol{a}_\theta|} = (-\sin\theta, \cos\theta, 0) \\
\boldsymbol{a}_z &= \frac{\partial \boldsymbol{r}}{\partial z} = (0, 0, 1) & \boldsymbol{e}_z &= \frac{\boldsymbol{a}_z}{|\boldsymbol{a}_z|} = (0, 0, 1)
\end{aligned}
$$

derivatives of unit vectors

$$
\begin{aligned}
\frac{\partial \boldsymbol{e}_r}{\partial \theta} &= (-\sin\theta, \cos\theta, 0) = \boldsymbol{e}_\theta \\
\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} &= (-\cos\theta, \sin\theta, 0) = -\boldsymbol{e}_r
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{e}}_r &= \frac{\partial \boldsymbol{e}_r}{\partial \theta}\dot{\theta} = \dot{\theta}\boldsymbol{e}_\theta \\
\dot{\boldsymbol{e}}_\theta &= \frac{\partial \boldsymbol{e}_\theta}{\partial \theta}\dot{\theta} = -\dot{\theta}\boldsymbol{e}_r
\end{aligned}
$$

position, velocity and acceleration vectors

$$
\begin{aligned}
\boldsymbol{r} &= r\boldsymbol{e}_r + z\boldsymbol{e}_z \\
\boldsymbol{v} &= \dot{\boldsymbol{r}} = (r\boldsymbol{e}_r + z\boldsymbol{e}_z)^{\cdot} = \dot{r}\boldsymbol{e}_r + r\underbrace{\dot{\boldsymbol{r}}}_{\dot{\theta}\boldsymbol{e}_\theta} + \dot{z}\boldsymbol{e}_z + z\underbrace{\dot{\boldsymbol{e}}_z}_{=\boldsymbol{0}} \\
&= \underbrace{\dot{r}}_{v_r}\boldsymbol{e}_r + \underbrace{r\dot{\theta}}_{v_\theta}\boldsymbol{e}_\theta + \underbrace{\dot{z}}_{v_z}\boldsymbol{e}_z \\
\boldsymbol{a} &= (\dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z)^{\cdot} \\
&= (\ddot{r}\boldsymbol{e}_r + \dot{r}\underbrace{\dot{\boldsymbol{e}}_r}_{\dot{\theta}\boldsymbol{e}_\theta} + \dot{r}\dot{\theta}\boldsymbol{e}_\theta + r\ddot{\theta}\boldsymbol{e}_\theta + r\dot{\theta}\underbrace{\dot{\boldsymbol{e}}_\theta}_{-\dot{\theta}\boldsymbol{e}_r} + \ddot{z}\boldsymbol{e}_z) \\
&= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\boldsymbol{e}_r + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{e_\theta}\boldsymbol{e}_\theta + \underbrace{\ddot{z}}_{a_z}\boldsymbol{e}_z
\end{aligned}
$$

Frenét coordinates

path - oriented curve with an origin O^*,

position (natural parameter) of a particle - $\overset{+}{-}$ distance from the particle to origin O^*, measured along the path,

No representation for the position vector.

Velocity vector

$$
\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \underbrace{\frac{d\boldsymbol{r}}{ds}}_{\boldsymbol{e}_t}\frac{ds}{dt} = \underbrace{\dot{s}}_{v}\boldsymbol{e}_t
$$

$\boldsymbol{e}_t$ - unit vector, tangent to the path, with orientation consistent with the path

Acceleration vector

$$
\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \ddot{s}\boldsymbol{e}_t + \dot{s}\dot{\boldsymbol{e}}_t = \ddot{s}\boldsymbol{e}_t + \dot{s}\frac{d\boldsymbol{e}_t}{ds}\dot{s}
$$

$\boldsymbol{e}_t^2 = 1$ implies $2\boldsymbol{e}_t \cdot \frac{d\boldsymbol{e}_t}{ds} = 0$, i.e. $\frac{d\boldsymbol{e}_t}{ds}$ is perpendicular to the tangent line.

Introducing:

- curvature $\mu = |\frac{d\boldsymbol{e}_t}{ds}|$,
- unit (principal) normal vector $\boldsymbol{e}_n = \frac{\frac{d\boldsymbol{e}_t}{ds}}{\mu}$,
- radius of curvature $\rho = \mu^{-1}$,

we get,

$$\boldsymbol{a} = \underbrace{\ddot{s}}_{a_t}\boldsymbol{e}_t + \underbrace{\frac{\dot{s}^2}{\rho}}_{a_n}\boldsymbol{e}_n$$

Vectors $\boldsymbol{e}_t, \boldsymbol{e}_n$, and binormal unit vector $\boldsymbol{e}_b = \boldsymbol{e}_t \times \boldsymbol{e}_n$ define Frenét system of coordinates. Velocity has only tangential component, and acceleration has only tangential and principal normal components.

For a planar curve $y = y(x)$,

$$\mu = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}}$$

2 Equations of motion

$$m\boldsymbol{a} = \boldsymbol{F}$$

Cartesian coordinates

$$\begin{aligned} ma_x &= m\ddot{x} &= F_x \\ ma_y &= m\ddot{y} &= F_y \\ ma_z &= m\ddot{z} &= F_z \end{aligned}$$

Cylindrical coordinates

$$\begin{aligned} ma_r = && m(\ddot{r} - r\dot{\theta}^2) &= F_r \\ ma_y = && m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= F_\theta \\ ma_z = && m\ddot{z} &= F_z \end{aligned}$$

Frenét coordinates

$$\begin{aligned} ma_t = && m\ddot{s} &= F_t \\ ma_n = && m\tfrac{\dot{s}^2}{\rho} &= F_n \\ && 0 &= F_b \end{aligned}$$

3 Principle of Work and Energy

Work of a force (field) $\boldsymbol{F}$ along curve AB

$$U_{12} = \int_{AB} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{u_1}^{u_2} \left(\boldsymbol{F} \cdot \frac{d\boldsymbol{R}}{du} \right) du$$

where $\boldsymbol{r} = \boldsymbol{r}(u), u_1 \leq u \leq u_2, \boldsymbol{r}(u_1) = A, \boldsymbol{r}(u_2) = B$ is a parameterization for the curve.

Single particle

$$a_t = \frac{dv}{dt}, \quad v = \frac{ds}{dt} \Rightarrow a_t ds = v dv$$

$$mv\,dv = \underbrace{ma_t}_{F_t}\,ds = F_t ds = \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$\int_{v_1}^{v_2} mv\,dv = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$\underbrace{\frac{mv_2^2}{2}}_{T_2} - \underbrace{\frac{mv_1^2}{2}}_{T_1} = \underbrace{\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r}}_{U_{12}}$$

$$T_1 + U_{12} = T_2$$

System of particles

$$\sum \frac{m_i v_{i,1}^2}{2} + \sum U_{i,12}^{\text{active}} + \sum U_{i,12}^{\text{reactive}} = \sum \frac{m_i v_{i,2}^2}{2}$$

For particles connected with inextensible cables or links, the work of reactive forces can be neglected.

Case of two particles connected with a rigid link

$$\boldsymbol{R}_A = -R\frac{\boldsymbol{AB}}{|\boldsymbol{AB}|}, \quad \boldsymbol{R}_B = R\frac{\boldsymbol{AB}}{|\boldsymbol{AB}|}$$

$$\boldsymbol{AB}^2 = (\boldsymbol{r}_B - \boldsymbol{r}_A)^2 = \text{const}$$

$$2\boldsymbol{AB} \cdot (d\boldsymbol{r}_B - d\boldsymbol{r}_A) = 0$$

$$\underbrace{R\frac{\boldsymbol{AB}}{|\boldsymbol{AB}|}}_{\boldsymbol{R}_B} \cdot d\boldsymbol{r}_B + \underbrace{\left(-R\frac{\boldsymbol{AB}}{|\boldsymbol{AB}|}) \right)}_{\boldsymbol{R}_A} \cdot d\boldsymbol{r}_A = 0$$

4 Conservative Forces

A force field $\boldsymbol{F}$ is conservative if the work done by $\boldsymbol{F}$ along *any* closed path is zero. If $\boldsymbol{F}$ is defined in a simply connected domain then the following conditions are equivalent to each other.

i $\boldsymbol{F}$ is conservative,

ii There exists a potential (energy) V such that

$$\boldsymbol{F} - -\boldsymbol{\nabla} V$$

iii $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$.

Cartesian coordinates

$$\begin{aligned}
\boldsymbol{\nabla} V &= \frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k} \\
\boldsymbol{\nabla} \times \boldsymbol{F} &= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)
\end{aligned}$$

Polar coordinates

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{e}_\theta$$

Examples of conservative forces

Weight

$$\boldsymbol{F} = (0, 0, -W)$$

$$\boldsymbol{V} = Wz + c$$

Gravitational force

$$\begin{aligned}
\boldsymbol{F} &= -\frac{GMm}{r^2}\frac{\boldsymbol{r}}{r} \\
\boldsymbol{V} &= \frac{GMm}{r} + c
\end{aligned}$$

Spring force

$$\begin{aligned}
\boldsymbol{F} &= -k\Delta l\frac{\boldsymbol{r}}{r} \\
\boldsymbol{V} &= \frac{k\Delta l^2}{2} + c
\end{aligned}$$

Here $\boldsymbol{r} = (x, y, z)$ is position vector, $r = |\boldsymbol{r}|$, and $\Delta = r - l_0$ is the stretch of the spring (l_0 denotes the unstretched length).

5 Principle of Linear Impulse and Momentum

Single particle

$$m\dot{\boldsymbol{v}} = \boldsymbol{F}$$

$$m\boldsymbol{v}_2 - m\boldsymbol{v}_1 = \int_{t_1}^{t_2} \boldsymbol{F}\, dt$$

$$\underbrace{m\boldsymbol{v}_1}_{\text{initial momentum}} + \underbrace{\int_{t_1}^{t_2} \boldsymbol{F}\, dt}_{\text{impulse}} = \underbrace{m\boldsymbol{v}_2}_{\text{final momentum}}$$

Frenét coordinates version

$$m\dot{v} = \boldsymbol{F}$$

$$mv_2 - mv_1 = \int_{t_1}^{t_2} F_t\, dt$$

$$\underbrace{mv_1}_{\text{initial momentum}} + \underbrace{\int_{t_1}^{t_2} F_t\, dt}_{\text{tangential impulse}} = \underbrace{mv_2}_{\text{final momentum}}$$

System of particles

$$\underbrace{\sum m_i \boldsymbol{v}_{i,1}}_{\text{initialmomentum}} + \underbrace{\int_{t_1}^{t_2} \boldsymbol{F}_i^{\text{active}}\, dt}_{\text{impulse}} = \underbrace{\sum m_i \boldsymbol{v}_{i,2}}_{\text{final momentum}}$$

6 Principle of Angular Impulse and Momentum

Single particle A. Point O fixed, $\boldsymbol{r} = \boldsymbol{OA}$

$$m\dot{\boldsymbol{v}} = \boldsymbol{F}$$

$$\boldsymbol{r} \times m\dot{\boldsymbol{v}} = \boldsymbol{r} \times \boldsymbol{F}$$

But

$$(\boldsymbol{r} \times m\boldsymbol{v})^{\cdot} = \underbrace{\dot{\boldsymbol{r}}}_{\boldsymbol{v}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} = \boldsymbol{r} \times m\boldsymbol{v}\ ,$$

so

$$\underbrace{(\boldsymbol{r} \times m\boldsymbol{v})^{\cdot}}_{\text{angular momentum } \boldsymbol{H}_O} = \underbrace{\boldsymbol{r} \times \boldsymbol{F}}_{\text{moment } \boldsymbol{M}_O}$$

$$\underbrace{\boldsymbol{r} \times m\boldsymbol{v}_1}_{\text{initial angular momentum}} + \underbrace{\int_{t_1}^{t_2} \boldsymbol{M}_O(t)\, dt}_{\text{angular impulse}} = \underbrace{\boldsymbol{r} \times m\boldsymbol{v}_2}_{\text{final angular momentum}}$$

System of particles. Point O fixed, $\boldsymbol{r}_i = \boldsymbol{OA}_i$

$$\underbrace{\left(\sum_i \boldsymbol{r}_i \times m_i\boldsymbol{v}_i\right)^{\cdot}}_{\boldsymbol{H}_O} = \underbrace{\sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i}_{\boldsymbol{M}_O}$$

$$\boldsymbol{H}_O^{\text{initial}} + \int_{t_1}^{t_2} \boldsymbol{M}_O(t)\, dt = \boldsymbol{H}_O^{\text{final}}$$

System of particles. Point O - fixed, C - center of mass

$$\begin{aligned}
\boldsymbol{H}_C &= \sum_i \boldsymbol{CA}_i \times m_i\boldsymbol{v}_i = \sum_i (\boldsymbol{CO} + \boldsymbol{OA}_i) \times m_i\boldsymbol{v}_i \\
&= \boldsymbol{CO} \times \sum_i m_i\boldsymbol{v}_i + \sum_i \boldsymbol{OA}_i \times m_i\boldsymbol{v}_i
\end{aligned}$$

From the definition of center of mass, $M = \sum_i m_i$ - total mass,

$$\begin{aligned}
\boldsymbol{OC} &= \frac{\sum_i m_i \boldsymbol{OA}_i}{M} \\
M\, \boldsymbol{OC} &= \sum_i m_i\, \boldsymbol{OA}_i \quad / \frac{d}{dt} \\
M\, \boldsymbol{v}_c &= \sum_i m_i\boldsymbol{v}_i
\end{aligned}$$

So

$$\begin{aligned}
\boldsymbol{H}_C &= \boldsymbol{CO} \times M\boldsymbol{v}_C + \boldsymbol{H}_O \quad / \frac{d}{dt} \\
\dot{\boldsymbol{H}}_C &= -\boldsymbol{v}_C \times M\boldsymbol{v}_C + \boldsymbol{CO} \times M\dot{\boldsymbol{v}}_C + \dot{\boldsymbol{H}}_O \\
&= \boldsymbol{CO} \times \sum_i \boldsymbol{F}_i + \sum_i \boldsymbol{OA}_i \times \boldsymbol{F}_i \\
&= \sum_i (\boldsymbol{CO} + \boldsymbol{OA}_i) \times \boldsymbol{F}_i \\
&= \sum_i \boldsymbol{CA}_i \times \boldsymbol{F}_i = \boldsymbol{M}_C
\end{aligned}$$

7 Kinematics of a Rigid Body

For arbitrary points P and A on the rigid body

$$\boldsymbol{v}_P = \boldsymbol{v}_A + \underbrace{\boldsymbol{\omega}}_{\text{angular velocity vector}} \times \boldsymbol{AP}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_A + \underbrace{\boldsymbol{\alpha}}_{\text{angular acceleration vector}} \times \boldsymbol{AP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{AP})$$

8 Moving Frame of Reference

The frame of reference is identified with a Cartesian system with origin at point A, angular velocity vector $\boldsymbol{\Omega}$, and unit vectors $\boldsymbol{e}_i$.

Time derivative of a vector-valued function $\boldsymbol{f}(t) = \sum_i f_i(t)\boldsymbol{e}_i(t)$.

$$\begin{aligned}
\dot{\boldsymbol{f}} &= \sum_i \dot{f}_i \boldsymbol{e}_i + \sum_i f_i \underbrace{\dot{\boldsymbol{e}}_i}_{\boldsymbol{\Omega}\times\boldsymbol{e}_i} \\
&= \sum_i \dot{f}_i \boldsymbol{e}_i + \Omega \times (\sum_i f_i \boldsymbol{e}_i) \\
&= \underbrace{\dot{\boldsymbol{f}}|_{\text{rel}}}_{\text{relative time derivative}} + \boldsymbol{\Omega} \times \boldsymbol{f}
\end{aligned}$$

Velocity and acceleration

$$\begin{aligned}
\boldsymbol{OP} &= \boldsymbol{OA} + \boldsymbol{AP} \\
\boldsymbol{v}_P &= \boldsymbol{v}_A + \underbrace{\dot{\boldsymbol{AP}}|_{\text{rel}}}_{\boldsymbol{v}_{P,\text{rel}}} + \boldsymbol{\Omega} \times \boldsymbol{AP} \\
\boldsymbol{a}_P &= \boldsymbol{a}_A + \underbrace{\dot{\boldsymbol{v}}_{P,\text{rel}}|_{\text{rel}}}_{\boldsymbol{a}_{P,\text{rel}}} + \boldsymbol{\Omega} \times \boldsymbol{v}_{P,\text{rel}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{AP} + \boldsymbol{\Omega} \times (\boldsymbol{v}_{P,\text{rel}} + \boldsymbol{\Omega} \times \boldsymbol{AP}) \\
&= \underbrace{\boldsymbol{a}_A + \dot{\boldsymbol{\Omega}} \times \boldsymbol{AP} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{AP})}_{\text{rigid body acceleration}} + \underbrace{\boldsymbol{a}_{P,\text{rel}}}_{\text{relative acceleration}} + \underbrace{2\boldsymbol{\Omega} \times \boldsymbol{v}_{P,\text{rel}}}_{\text{Coriolis acceleration}}
\end{aligned}$$

9 Rigid Body Kinetics

Tensor(matrix) of inertia. For O fixed, or $O = C$, center of mass,

$$\boldsymbol{H}_O = \underbrace{\boldsymbol{I}_O}_{\text{inertia tensor}} \boldsymbol{\omega}$$

Equation of rotational motion for a rigid body. For O fixed, or $O = C$, center of mass,

$$\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O \quad \text{implies} \quad (\boldsymbol{I}_O\boldsymbol{\omega})^{\cdot} = \boldsymbol{M}_O$$

In a system of coordinates rotating with angular velocity vector $\boldsymbol{\Omega}$, in which $\dot{\boldsymbol{I}}_O = \boldsymbol{0}$,

$$\boldsymbol{I}_O\dot{\boldsymbol{\omega}}|_{\text{rel}} + \boldsymbol{\Omega} \times (\boldsymbol{I}_O\boldsymbol{\omega}) = \boldsymbol{M}_O$$

In 2D, $\boldsymbol{I}_O\boldsymbol{\omega} = (0, 0, I_O\omega)$, $\quad \boldsymbol{M}_O = (0, 0, M_O)$ imply

$$I_O\alpha = M_O$$

10 Kinetic Energy of a Rigid Body

Arbitrary motion

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i(\boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{CP}_i)^2 \\
&= \frac{1}{2}\sum_i m_i\left(\boldsymbol{v}_C^2 + 2\boldsymbol{v}_C \circ (\boldsymbol{\omega} \times \boldsymbol{CP}_i) + (\boldsymbol{\omega} \times \boldsymbol{CP}_i)^2\right) \\
&= \frac{1}{2}\sum_i m_i v_C^2 + \boldsymbol{v}_C \circ \boldsymbol{\omega} \times \underbrace{\sum_i m_i \boldsymbol{CP}_i}_{\boldsymbol{0}} + \frac{1}{2}\sum_i m_i(\boldsymbol{\omega} \times \boldsymbol{CP}_i)^2
\end{aligned}$$

In 2D,

$$(\boldsymbol{\omega} \times \boldsymbol{CP}_i)^2 = (x_i^2 + y_i^2)\omega^2$$

so,

$$\sum_i m_i(\boldsymbol{\omega} \times \boldsymbol{CP}_i)^2 = \underbrace{\sum_i m_i(x_i^2 + y_i^2)}_{I_C}\,\omega^2$$

and,

$$T = \frac{1}{2}Mv_c^2 + \frac{1}{2}I_C\omega^2$$

Rotation about a fixed point O, $d = |\boldsymbol{OC}|$.

$$\begin{aligned} T &= \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2 \\ &= \frac{1}{2} M (\omega d)^2 + \frac{1}{2} I_C \omega^2 \\ &= \frac{1}{2} \underbrace{(M d^2 + I_C)}_{I_O} \omega^2 \end{aligned}$$