CSE386L MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
Spring 22, Final Exam

1. Characteristics. Consider the equation:

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0\,.$$

Use the method of prime integrals to find the general solution of the equation. Complement then the equation with the initial condition:

$$u(x, y, 0) = x + 2y\,,$$

and find the solution to the initial-value problem. Use characteristics to verify the solution.

(20 points)

Solution: Equation for characteristics:

$$\frac{dx}{dt} = 3 \quad \frac{dy}{dt} = 2 \quad \frac{dz}{dt} = 1 \quad \text{or} \quad \frac{dx}{3} = \frac{dy}{2} = dz\,.$$

Prime integrals:

$$\begin{aligned} dx = 3z \qquad &\Rightarrow \quad x - 3z = C_1 \\ dy = 2z \qquad &\Rightarrow \quad y - 2z = C_2\,. \end{aligned}$$

General solution:

$$u = F(C_1, C_2) = F(x - 3z, y - 2z)\,.$$

Accounting for IC at $z = 0$:

$$F(x, y) = x + 2y\,.$$

Final solution of the IVB:

$$u = x - 3z + 2(y - 2z)\,.$$

Alternatively, the parametric equation for a characteristics emanating from (x_0, y_0, z_0) is:

$$x = 3t + x_0, \quad y = 2t + y_0, \quad z = t + z_0\,.$$

Intersection with plane $z = 0$ gives: $t = -z_0$, and $x = x_0 - 3z_0, y = y_0 - 2z_0$. Consequently,

$$u(x_0, y_0, z_0) = x + 2y = x_0 - 3z_0, +2(y_0 - 2z_0)\,,$$

the same solution as above.

2. Calculus of variations. Given a point P and a line l in plane $\mathbb{R}^2$, see Fig. 1, determine the shortest curve connecting the point with the line. Write down *precisely* the minimization problem, the corresponding variational formulation, and the E-L BVP. Solve then the BVP.

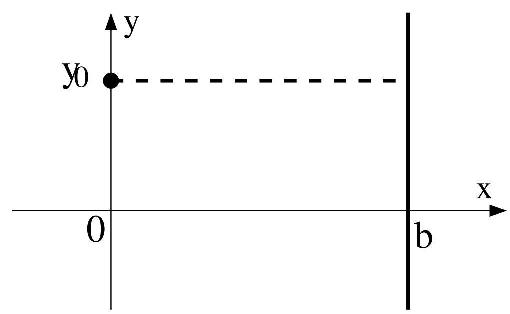

Figure 1: The shortest path between a point and a line.

(20 points)

Solution: See the lecture notes.

3. Jordan decomposition and systems of ODEs. Consider the matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

- Determine generalized eigenvectors of matrix A and the corresponding Jordan form.
- Use the Jordan form to determine general solution for the system of ODEs:

$$\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u}\,.$$

(20 points)

Solution:

- We have a single eigenvalue $\lambda = 1$, and a double eigenvalue $\lambda = 2$.
  Determining the eigenvector for $\lambda = 1$.

$$(\boldsymbol{A} - \lambda I)x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \Rightarrow \quad x = (t, 0, 0)^T\,.$$

  Choose $t = 1$.
  Determining an eigenvector for $\lambda = 2$.

$$(\boldsymbol{A} - \lambda I)x = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \Rightarrow \quad x = (t, t, 0)^T\,.$$

  The corresponding generalized eigenvector:

$$(\boldsymbol{A} - \lambda I)x = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \quad \Rightarrow \quad x = (s, t + s, 0)^T\,.$$

  Choosing $t = s = 1$, we obtain vectors:

$$a_1 = (1, 0, 0)^T, \quad a_2 = (1, 1, 0)^T, \quad a_3 = (1, 2, 0)^T\,.$$

  In the system of coordinates generated by the vectors, the matrix takes the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Seeking the solution in the form: $x = y_i(t)a_i$, we obtain the system of equations:

$$\begin{aligned} &\dot{y}_1 = y_1 && \Rightarrow y_1 = C_1 e^t \\ &\dot{y}_2 = 2y_2 + y_3 && \qquad \Rightarrow \dot{y}_2 - 2y_2 = C_3 e^{2t} \Rightarrow y_2 = (C_3 t + C_2)e^{2t} \\ &\dot{y}_3 = 2y_3 && \Rightarrow y_3 = C_3 e^{2t} \end{aligned}$$

The final solution is thus:

$$x = C_1 e^t a_1 + (C_3 t + C_2)e^{2t} a_2 + C_3 e^{2t} a_3 \,.$$

4. Separation of variables. Consider the BVP defined in Fig. 2. Use separation of variables to derive the solution in a form of a series. Define the coefficients in the series but you do not need to compute them.

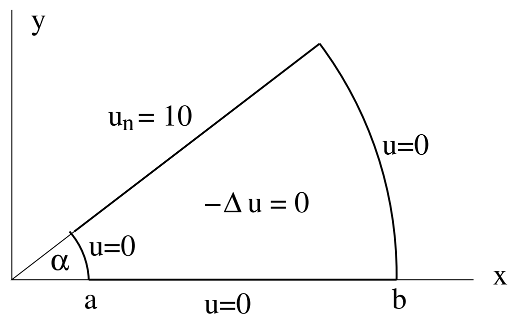

Figure 2: Laplace equation in a conical domain. u_n stands for the normal derivative $\frac{\partial u}{\partial n}$.

(20 points)

Solution 1: Use separation of variables: $u = R(r)\Theta(\theta)$ to arrive at:

$$-\frac{1}{r}(rR')'\Theta - \frac{1}{r^2}R\Theta'' = 0 \quad \Rightarrow \quad -\frac{r(rR')'}{R} = \frac{\Theta''}{\Theta} = \lambda\,.$$

BCs imply that we should consider the Sturm-Liouville problem in r,

$$LR := -r(rR')' = \lambda R\,.$$

The operator L is self-adjoint in the weighted space $L^1_{1/r}$ and positive definite (easy to check) so $\lambda > 0$. Assume $\lambda = k^2, k > 0$. We obtain the Cauchy-Euler equation:

$$-r^2R'' - rR' - k^2 = 0\,.$$

Seeking $R = r^\alpha$, we obtain $\alpha = \pm ik$. This gives

$$R = r^{\pm ik} = e^{\pm ik\ln r} = \cos(k\ln r) \pm i\sin(k\ln r)\,.$$

Alternatively, we can represent the general solution as:

$$R = C\cos(k\ln r) + D\sin(k\ln r)\,.$$

The condition for the existence of non-trivial solutions with the homogeneous BCs is:

$$\begin{vmatrix} \cos(k\ln a) & \sin(k\ln a) \\ \cos(k\ln b) & \sin(k\ln b) \end{vmatrix} = \sin(k\ln\frac{b}{a}) = 0 \quad \Rightarrow \quad k = k_n = \frac{n\pi}{\ln\frac{b}{a}}\,.$$

With known separation constant, the corresponding Θ is:

$$\Theta = A_n e^{k_n\theta} + B_n e^{-k_n\theta} \,,$$

or,

$$\Theta = A_n \cosh(k_n\theta) + B_n \sinh(k_n\theta) \,.$$

Imposing the BC at $\theta = 0$, we get $A_n = 0$. The ultimate solution is:

$$u = \sum_{n=1}^{\infty} \underbrace{\left(\cos(k_n \ln r) - \frac{\cos(k_n \ln a)}{\sin(k_n a)} \sin(k_n \ln r)\right)}_{=:R_n(r)} B_n \sinh(k_n(\theta)) \,.$$

Constants B_n are determined from the BC for $\theta = \alpha$,

$$\sum_{n=1}^{\infty} R_n(r) B_n k_n \cosh(k_n(\alpha)) = 10 \,.$$

Multiplying both sides with $\frac{1}{r} R_m(r)$, integrating over (a, b) interval, and using the $L^2_{1/r}$-orthogonality of functions $R_n(r)$, we get

$$B_m = -\frac{\int_a^b \frac{10}{r} R_m(r)\, dr}{\int_a^b \frac{1}{r} R_m^2(r)\, dr\, k_n \cosh(k_n\alpha)} \,.$$

Solution 2: As the Sturm-Liouville problem in r is more complicated than in θ, we may try to reverse the role of r and θ by introducing a *lift* U of Neumann data $u_n = 10$ at $\theta = \alpha$ such that U satisfies the Laplace equation. Upon inspection, we propose

$$U = 10\,\theta \,.$$

Indeed, $U = 0$ at $\theta = 0$ and $\frac{\partial U}{\partial n} = \frac{\partial U}{\partial \theta} = 10$ everywhere, also at $\theta = \alpha$. Function U is linear in θ and independent of r, so it satisfies the Laplace equation. We seek the ultimate solution in the form

$$u = U + v$$

where

$$v = -U \quad \Rightarrow \quad v = -U \text{ at } r = a, b \text{ and } v = 0 \text{ at } \theta = 0, \alpha \,.$$

We have now a Sturm-Liouville problem in θ:

$$-\Theta'' = \lambda\Theta, \quad \Theta(0) = 0,\ \Theta'(\alpha) = 0 \,.$$

As the operator is self-adjoint and positive-definite (easy to check), we can look for the separation constant $\lambda = k^2$. We obtain:

$$\Theta = A \cos k\theta + B \sin k\theta \,.$$

$\Theta(0) = 0$ implies $A = 0$, and $\Theta'(\alpha)$ implies

$$\cos k\alpha = 0 \quad k = k_n = \frac{1}{\alpha}(\frac{\pi}{2} + k\pi) \quad k = 1, 2, \ldots$$

The solution of the corresponding Cauchy-Euler equation is:

$$R_n = A_n r^{k_n} + B_n r^{-k_n}\,,$$

which gives:

$$v = \sum_{n=1}^{\infty} (A_n r^{k_n} + B_n r^{-k_n}) \sin(k_n\theta)\,.$$

Enforcing BCs at $r = a$ and $r = b$, we get:

$$\sum_{n=1}^{\infty} (A_n a^{k_n} + B_n a^{-k_n}) \sin(k_n\theta) = -10\,\theta \quad \text{and} \quad \sum_{n=1}^{\infty} (A_n b^{k_n} + B_n b^{-k_n}) \sin(k_n\theta) = -10\,\theta\,.$$

Multiplying both equations with $\sin k_m\theta$, integrating over $(0, \alpha)$, utilizing the L^2-orthogonality of eigenfunctions $\sin k_n\theta$, and noticing that

$$\int_0^{\alpha} \sin^2(k_m\theta)\, d\theta = \int_0^{\alpha} \frac{1 - \cos 2k_m\theta}{2}\, d\theta = \frac{\alpha}{2}\,,$$

we get a system of two equations for constants A_n, B_n,

$$\begin{pmatrix} a^{k_n} & a^{-k_n} \\ b^{k_n} & b^{-k_n} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = -10 \int_0^{\alpha} \theta \sin k_n\theta\, d\theta \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$$

5. Variational formulations. Consider the bar problem shown in Fig. 3. Formulate the energy minimization problem and the corresponding variational formulation. Discuss the equivalence of the two problems and prove the well-posedness of the variational formulation.

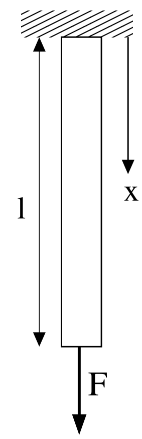

Figure 3: Elastic bar with stiffness EA, loaded with a force F.

(20 points)

Solution:

Total potential energy:

$$J(u) = \frac{1}{2}b(u,u) - l(u)$$

where:

$$b(u,v) = EA\int_0^l u'v'\,dx \qquad l(v) = Pv(l)\,.$$

Energy space:

$$V := \{u \in H^1(0,l) \ : \ u(0) = 0\}\,.$$

Minimization problem:

$$\begin{cases} \text{Find } u \in V \text{ such that:} \\ J(u) = \min_{w\in V} J(w)\,. \end{cases}$$

The corresponding variational formulation:

$$\begin{cases} \text{Find } u \in V \text{ such that :} \\ b(u,v) = l(v) \quad \forall\, v \in V\,. \end{cases}$$

The problems are equivalent since form $b(u,v)$ is symmetric and positive-definite.

In order to prove the well-posedness, we need to check:

Continuity of the bilinear form:

$$|b()u,v)| = |EA\int_0^l u'v'\,dx| \le EA\|u'\|\,\|v'\| \le EA\|u\|_{H^1}\,\|v\|_{H^1}\,.$$

Continuity of the linear form:

$$|l(v)| \le P|v(1)| \le PC\|v\|_{H^1},$$

where C is the continuity constant of the embedding $H^1(0,l) \hookrightarrow C([0,l])$.

Completness of the energy space: As the operation $H^1(0,l) \ni v \to v(0) \in \mathbb{R}$ is continuous, V is a closed subspace of Hilbert space $H^1(0,l)$ and, therefore, it is itself a Hilbert space.

V**-Coercivity of the bilinear form:** This follows form the Poincaré inequality:

$$\int_0^l (v')^2\,dx \ge c_P\|v\|^2 \quad v \in V\,.$$

Indeed,

$$\begin{aligned} \tfrac{1}{EA}b(v,v) &= \|v'\|^2 \\ \tfrac{1}{EAc_P}b(v,v) &\ge \|v\|^2 \end{aligned}$$

Summing up,

$$\frac{1}{EA}(1+c_P^{-1})b(v,v) \ge \|v\|^2_{H^1} \quad \Rightarrow \quad b(v,v) \ge EA(1+c_P^{-1})^{-1}\|v\|^2_{H^1} \quad v \in V\,.$$